SCANDINAVIAN ACTUARIAL JOURNAL, 2018
VOL. 2018, NO. 8, 727–752
https://doi.org/10.1080/03461238.2018.1452285

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Moments of renewal shot-noise processes and their applications

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ABSTRACT
In this paper, we study the family of renewal shot-noise processes. The Feynmann–Kac formula is obtained based on the piecewise deterministic Markov process theory and the martingale methodology. We then derive the Laplace transforms of the conditional moments and asymptotic moments of the processes. In general, by inverting the Laplace transforms, the asymptotic moments and the first conditional moments can be derived explicitly; however, other conditional moments may need to be estimated numerically. As an example, we develop a very efficient and general algorithm of Monte Carlo exact simulation for estimating the second conditional moments. The results can be then easily transformed to the counterparts of discounted aggregate claims for insurance applications, and we apply the first two conditional moments for the actuarial net premium calculation. Similarly, they can also be applied to credit risk and reliability modelling. Numerical examples with four distribution choices for interarrival times are provided to illustrate how the models can be implemented.

ARTICLE HISTORY
Received 16 March 2017
Accepted 11 March 2018

KEYWORDS
Renewal shot-noise processes; discounted aggregate claims; actuarial net premium; piecewise-deterministic Markov processes; martingale method; Monte Carlo exact simulation; credit risk; reliability

1. Introduction

Since the beginning of the twentieth century, *shot-noise processes* have been extensively used to model a very wide variety of natural phenomena, with numerous applications in electronics, optics, biology and many other fields in natural science, see early literature in Campbell (1909a, 1909b), Schottky (1918), Picinbono *et al.* (1970) and Verveen & DeFelice (1974). More recent applications extended to insurance and actuarial science in particular can be found in Klüppelberg & Mikosch (1995), Brémaud (2000), Dassios & Jang (2003, 2005), Jang (2004), Jang & Krvavych (2004), Torrisi (2004), Albrecher & Asmussen (2006), Macci & Torrisi (2011), Zhu (2013) and Schmidt (2014). Mostly, they adopted the classical *Poisson shot-noise process* (Cox & Isham 1980, p. 88), where the arrivals of claims are simply assumed to follow a Poisson process. However, an exponential distribution could be not appropriate for modelling claim interarrival times in practice when the likelihood of a claim given the time elapsed since the previous one is not constant over time. There has been a significant volume of literature that questions the appropriateness of a Poisson process in insurance modelling (Seal 1983, Beard *et al.* 1984) such as the rainfall modelling (Cox & Isham 1980, Smith 1980). For catastrophic events (e.g. floods, storms, hails, bushfires, earthquakes and terrorist attacks), the assumption that resulting claims occur in terms of a Poisson process is inadequate as it has a deterministic intensity, i.e. it has the same claim frequency rate between the same time interval of duration.

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A natural generalisation of Poisson process is the family of renewal processes (Cox 1962, Cox & Miller 1965, Karlin & Taylor 1975, Grandell 1991, Ross 1996, Rolski *et al.* 2008), which could offer more flexible model choices and are versatile enough to capture different styles of claim interarrival times in reality. Using ordinary, delayed and stationary renewal processes to derive the moments and moment generating functions of compound renewal sums with discounted claims can be found in Léveillé & Garrido (2001a, 2001b) and Léveillé *et al.* (2010). Since Andersen (1957) proposed to use a compound renewal risk model and Gerber & Shiu (1998) introduced the so-called discounted penalty function, the delayed and stationary renewal risk models and their extensions for modelling insurers' surplus processes can also be noticed in Willmot & Dickson (2003), Gerber & Shiu (2005), Li & Garrido (2005), Willmot (2007) and Woo (2010).

In this paper, we mainly study *renewal shot-noise processes*, the generalised family of Poisson shot-noise process. They are shot-noise processes driven by ordinary renewal processes, so that the interarrival times could be any positive independent identically distributed random variables. This paper can be considered as the generalisation of Jang (2004) from the classical Poisson process to a rather general renewal process for the underlying point process. However, this generalisation is technically nontrivial, since the renewal components lead our new models beyond the affine framework in general, and several new approaches have been adopted or developed to investigate the properties of moments. Based on the *piecewise deterministic Markov process theory* (Davis 1984, 1993) and the martingale methodology (Dassios & Embrechts 1989), we first obtain the Feynmann–Kac formula. We then derive the Laplace transforms of the conditional moments and asymptotic moments of the processes. In general, by inverting the Laplace transforms of these moments, any asymptotic moments as well as the first conditional moments can be derived explicitly, however, other conditional moments may need to be estimated numerically. As an example, we develop a very efficient and general algorithm of Monte Carlo exact simulation for estimating the second conditional moments. The results then can be easily transformed to the counterparts of discounted aggregate claims in insurance, and we apply the first two conditional moments for the actuarial net premium calculation. Similarly, they can also be applied to credit risk and reliability modelling. Numerical examples with four different distributions for modelling interarrival times are provided, and the implementation details are also discussed.

This paper is structured as follows. Section 2 introduces renewal shot-noise processes and the associated processes of discounted aggregate claims in insurance. In Section 3, based on the piecewise deterministic Markov process theory and the martingale methodology, we present the Feynmann–Kac formula. It is then used in Section 4 to derive the Laplace transforms of the moments of renewal shot-noise processes and discounted aggregate claims. Afterwards, in Section 5, we apply the results of the means and variances to the actuarial context for calculating net insurance premiums as well as credit risk and reliability modelling, for which we specify exponential, gamma, inverse Gaussian and folded normal distributions for modelling interarrival times, respectively. Section 6 contains concluding remarks.

2. Renewal shot-noise processes and discounted aggregate claims

Claims arising from catastrophic events could be different from the different time interval of duration, and they could also depend on the time elapsed since the previous claim. Therefore, improved models beyond the Poisson process to predict claims arising from catastrophic events are required. For this purpose, let us start with a compound model of insurance risk with the additional economic assumption of a positive interest rate, and the accumulated value of aggregate claims up to time t in continuous time on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is

$$L_t = \sum_{i=1}^{N_t} X_i e^{r\left(t-T_i\right)}, \qquad t \geq 0,$$

where

- $r > 0$ is the risk-free force of interest rate;
- $\{X_i\}_{i=1,2,\ldots}$ are claim sizes (or jump sizes), which are assumed to be independent and identically distributed (*i.i.d.*) with cumulative distribution function (CDF) $H(x), x > 0$;
- $\{T_i\}_{i=1,2,\ldots}$ are the claim occurrence times (or, *renewal epochs*), which follows a renewal point process $N_t = \sum_i \mathbf{1}_{\{T_i \le t\}}$ with $N_0 = 0$.

$\mathcal{F}_t$ is the associated natural filtration of L_t. Setting $L_t^0 = e^{-rt} L_t$, we have the discounted value at time 0 of aggregate claims (or, *discounted aggregate claims*) up to time t, i.e.

$$L_t^0 = \sum_{i=1}^{N_t} X_i e^{-rT_i}. \tag{2.1}$$

As Jang (2004) and Jang & Krvavych (2004) noted the duality property between the process of discounted aggregate claims and the shot-noise process, we now introduce a *renewal shot-noise process* (or, shot-noise process driven by an ordinary renewal process)

$$S_t = S_0 e^{-\delta t} + \sum_{i=1}^{N_t} X_i e^{-\delta(t-T_i)}, \tag{2.2}$$

where δ is a constant. Setting $S_0 = 0$ and $\delta = -r$ in (2.2), the processes of S_t and L_t become identical. S_t was also discussed in Rice (1977) and was used as the stochastic intensity of a *double stochastic Poisson process* (or *Cox process*) in Møller & Torrisi (2005) and Dassios *et al.* (2015). Simulated sample paths of the renewal shot-noise process S_t and the underlying renewal process N_t are provided in Figure 1, where we assume the interarrival times follow an inverse Gaussian distribution and jump sizes follow an exponential distribution.

Note that, this process S_t is no longer within the usual framework of *affine processes* (Duffie *et al.* 2000, 2003) or a Markov process due to the additional renewal components. In order to establish a Markovian framework, we need to further include a supplementary stochastic process U_t, the *backward recurrence time* (Cox 1962, p. 27) (or *the time elapsed since the last jump arrived*) in the process S_t, i.e.

$$U_t := U_0 + t - \sum_{i=1}^{N_t} \tau_i,$$

where $U_0 \ge 0$ is the initial value of U_t; $\{\tau_i\}_{i=1,2,\ldots}$ are interarrival times of claim arrivals, i.e.

$$\tau_i := T_i - T_{i-1}, \qquad i = 1, 2, \ldots, \qquad T_0 = 0,$$

and they are *i.i.d.* with the CDF $P(u), u > 0$, which is assumed to be absolutely continuous with the associated density function $p(u)$. The idea of adding this supplementary variable U_t to make the underlying process Markovian can be found as early as in Cox (1955). U_t increases at unit rate till a jump arrives, then it goes back to 0. Note that, if $\rho(u)$ is denoted as the *failure rate* of the distribution, we have

$$P(u) = 1 - \exp\left(-\int_0^u \rho(v)\mathrm{d}v\right), \qquad p(u) = \rho(u)\exp\left(-\int_0^u \rho(v)\mathrm{d}v\right),$$

where $\rho(u) = \frac{p(u)}{\bar{P}(u)}$, and $\bar{P}(u) := 1 - P(u)$ is denoted as the tail probability or the *survivor function* (Cox 1962, p. 3). For notation simplification, we denote the first mean and the Laplace transform respectively by

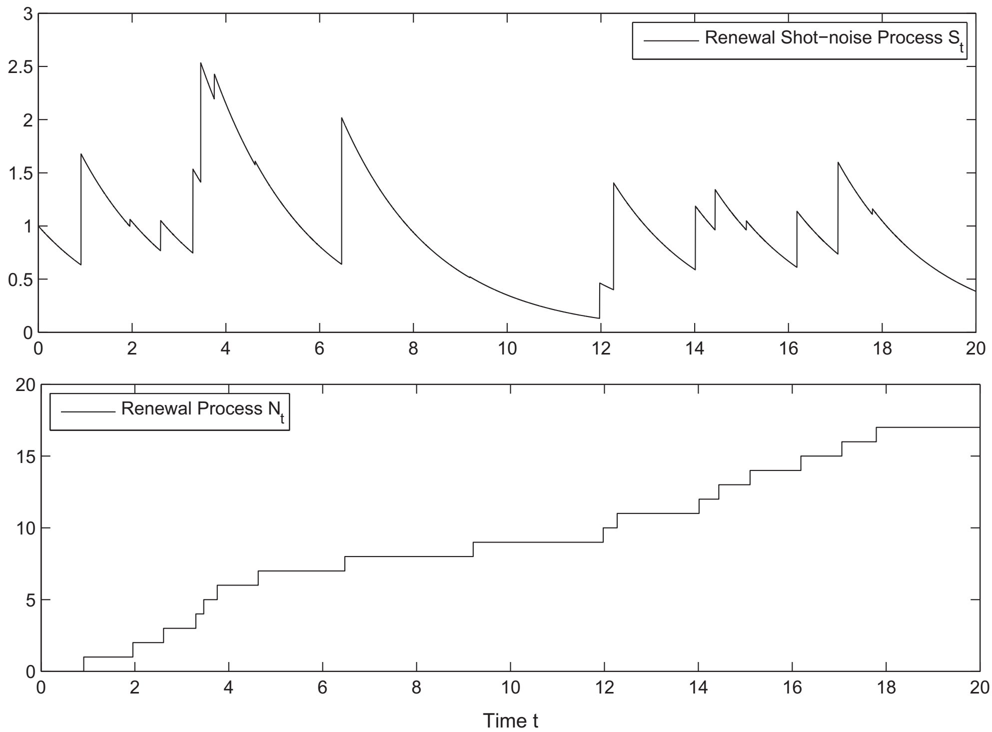

Figure 1. Simulated paths of renewal shot-noise process S_t and renewal process N_t when the interarrival times follow an inverse Gaussian distribution and jump sizes follow an exponential distribution.

$$\gamma_1 := \int_0^\infty u p(u)\mathrm{d}u < \infty, \qquad \hat{p}(\theta) := \int_0^\infty e^{-\theta u} p(u)\mathrm{d}u < \infty.$$

We denote the mth moment of S_t conditional on S_0 and U_0 and the associated Laplace transform with respect to time t respectively by

$$e_m(t; S_0, U_0; \delta) := \mathbb{E}\big[S_t^m \mid S_0, U_0\big], \qquad m \in \mathbb{N}^+,$$
$$\hat{e}_m(\theta; S_0, U_0; \delta) := \int_0^\infty e^{-\theta t}\mathbb{E}\big[S_t^m \mid S_0, U_0\big]\mathrm{d}t,$$

and the moments of claim amounts by

$$\mu_k := \int_0^\infty x^k \mathrm{d}H(x), \qquad k = 0, 1, 2, \ldots.$$

The Laplace transform of any given function $f(t)$ in general is denoted by

$$\hat{f}(t) := \mathcal{L}_\theta\big\{f(t)\big\} := \int_0^\infty e^{-\theta t} f(t)\mathrm{d}t.$$

All moments and Laplace transforms above are assumed to be finite.

3. Martingales

Let us define a process

$$Z_t = \int_0^t e^{-\theta u} \sum_{k=1}^m \kappa_k S_u^k \mathrm{d}u, \tag{3.1}$$

where $\theta \ge 0$ and $\{\kappa_k\}_{k=1,2,\ldots,m}$ are all constants. The *infinitesimal generator* of (Z_t, S_t, U_t, t) acting on any function $g\,(z, s, u, t)$ belonging to its domain is given by

$$\begin{aligned} \mathcal{A}g(z,s,u,t) = {} & \left(e^{-\theta t} \sum_{k=1}^m \kappa_k s^k\right) \frac{\partial g}{\partial z} + \frac{\partial g}{\partial t} + \frac{\partial g}{\partial u} - \delta s \frac{\partial g}{\partial s} \\ & + \frac{p(u)}{\bar{P}(u)} \left[\int_0^\infty g(z, s+x, 0, t)\mathrm{d}H(x) - g(z,s,u,t)\right], \end{aligned} \tag{3.2}$$

where $g : (0,\infty) \times (0,\infty) \times (0,\infty) \times \mathbb{R}^+ \to (0,\infty)$. It is sufficient that $g(z,s,u,t)$ is differentiable with respect to z, s, u, t for any z, s, u, t and that

$$\left| \int_0^\infty g(\cdot, s+x, \cdot, \cdot)\mathrm{d}H(x) - g(\cdot, s, \cdot, \cdot) \right| < \infty \tag{3.3}$$

for $g(z,s,u,t)$ to belong to the domain of the (extended) generator $\mathcal{A}$. For the details on finding the generator of (Z_t, S_t, U_t, t) using the *piecewise deterministic Markov process theory* (Davis 1984, 1993), see Dassios & Embrechts (1989), Dassios & Jang (2003), Rolski *et al.* (2008), Dassios & Zhao (2011, 2012, 2014) and many others.

Let us first provide a proposition as below which will be used very often in this paper.

Proposition 3.1: *The ordinary differential equation (ODE) of $A(u)$,*

$$a - \xi A(u) + A'(u) + \frac{p(u)}{\bar{P}(u)}\big[b + A(0) - A(u)\big] = 0, \tag{3.4}$$

has the solution

$$A(u) := \frac{a}{\xi} + \frac{b}{1-\hat{p}(\xi)} \frac{e^{\xi u}}{\bar{P}(u)} \int_u^\infty e^{-\xi v} p(v)\mathrm{d}v, \tag{3.5}$$

where a, b, ξ are all constants, and $\xi \ge 0$.

Now, in order to derive the mth moment of S_t conditional on S_0 and U_0 at time $t=0$ in the next section, we have to first find a suitable martingale with respect to the filtration $\mathcal{F}_t$, which is given in Theorem 3.1.

Theorem 3.1: *We have a $\mathcal{F}_t$-martingale*

$$Z_t + e^{-\theta t} \sum_{k=0}^m S_t^k A_k(U_t), \tag{3.6}$$

where

$$A_k(u) := \frac{\kappa_k}{\theta + \delta k} + \frac{\sum_{n=k+1}^m A_n(0)\binom{n}{k}\mu_{n-k}}{1 - \hat{p}(\theta+\delta k)} \frac{e^{(\theta+\delta k)u}}{\bar{P}(u)} \int_u^\infty e^{-(\theta+\delta k)v} p(v)\mathrm{d}v, \qquad k = 0, 1, \ldots, m-1, \tag{3.7}$$

and

$$A_m(u) := \frac{\kappa_m}{\theta + \delta m}. \tag{3.8}$$

Proof: To find a $\mathcal{F}_t$-martingale, we assume a function in form of

$$g(z,s,u,t) = z + e^{-\theta t}\sum_{k=0}^{m} s^k A_k(u). \tag{3.9}$$

Setting $\mathcal{A}g = 0$ in (3.2), we obtain the equation

$$\sum_{k=1}^{m}\kappa_k s^k - \theta\sum_{k=0}^{m} s^k A_k(u) + \sum_{k=0}^{m} s^k A_k'(u) - \delta\sum_{k=0}^{m} k s^k A_k(u) \\ + \frac{p(u)}{\bar{P}(u)}\left[\int_0^\infty \sum_{k=0}^{m}(s+x)^k A_k(0)\mathrm{d}H(x) - \sum_{k=0}^{m} s^k A_k(u)\right] = 0.$$

Note that, based on $(s+x)^k = \sum_{j=0}^{k}\binom{k}{j}s^j x^{k-j}$ where

$$\binom{k}{j} := \frac{k!}{j!(k-j)!}, \qquad j = 0,1,\ldots,k,$$

we have

$$0 = \sum_{k=1}^{m}\kappa_k s^k - \theta\sum_{k=0}^{m} s^k A_k(u) + \sum_{k=0}^{m} s^k A_k'(u) - \delta\sum_{k=0}^{m} k s^k A_k(u) \\ + \frac{p(u)}{\bar{P}(u)}\left[\int_0^\infty \sum_{k=0}^{m}\sum_{j=0}^{k}\binom{k}{j}s^j x^{k-j} A_k(0)\mathrm{d}H(x) - \sum_{k=0}^{m} s^k A_k(u)\right] \\ = \sum_{k=1}^{m}\kappa_k s^k - \theta\sum_{k=0}^{m} s^k A_k(u) + \sum_{k=0}^{m} s^k A_k'(u) - \delta\sum_{k=0}^{m} k s^k A_k(u) \\ + \frac{p(u)}{\bar{P}(u)}\left[\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{k}{j}s^j \mu_{k-j} A_k(0) - \sum_{k=0}^{m} s^k A_k(u)\right],$$

where

$$\mu_{k-j} = \int_0^\infty x^{k-j}\mathrm{d}H(x), \qquad j = 0,1,2,\ldots,k.$$

Then, setting $\kappa_0 = 0$, we can rewrite it by

$$\sum_{k=0}^{m} s^k\left[\kappa_k - (\theta+\delta k)A_k(u) + A_k'(u)\right] + \frac{p(u)}{\bar{P}(u)}\sum_{k=0}^{m}\left[A_k(0)\sum_{j=0}^{k}\binom{k}{j}\mu_{k-j}s^j - s^k A_k(u)\right] \\ = \sum_{k=0}^{m} c_k(u)s^k = 0, \tag{3.10}$$

where $c_k(u)$ is the coefficient of s^k, i.e.

$$c_k(u) := \kappa_k - (\theta+\delta k)A_k(u) + A_k'(u) + \frac{p(u)}{\bar{P}(u)}\left[\sum_{n=k}^{m} A_n(0)\binom{n}{k}\mu_{n-k} - A_k(u)\right], \quad k = 0,1,\ldots,m.$$

Since (3.10) should hold for any s^k where $\forall k \in \{0, 1, 2, \dots, m\}$, each coefficient should be equal to zero, i.e. we have the ODEs

$$c_k(u) = 0, \qquad k = 0, 1, \dots, m.$$

Using Proposition 3.1, we have the solutions

$$A_k(u) = \frac{\kappa_k}{\theta + \delta k} + \frac{\sum_{n=k}^{m} A_n(0)\binom{n}{k}\mu_{n-k}}{1 - \hat{p}(\theta + \delta k)} \frac{e^{(\theta+\delta k)u}}{\bar{P}(u)} \int_u^\infty e^{-(\theta+\delta k)v} p(v)\mathrm{d}v,$$

with the boundary conditions $A_k(0) = 0$ for $k = 0, 1, \dots, m$. More specifically, they are equivalent to (3.7) for $k = 0, 1, 2, \dots, m-1$ and (3.8) for $k = m$. Finally, it is easy to see that, this function (3.9) is differentiable with respect to all its arguments z, s, u, t, and also the expectation of the associated jumps is bounded, i.e. it satisfies (3.3). Hence, it belongs to the domain of the (extended) generator $\mathcal{A}$. It is based on the *piecewise deterministic Markov process theory*, which was developed by Davis (1984, Theorem 5.5, p. 367), see also more details on this theory and its conditions in the book by Davis (1993, p. 69). □

Proposition 3.2: *We have the Feynmann–Kac formula*

$$\mathbb{E}\left[\int_0^\infty e^{-\theta t} \sum_{k=1}^{m} \kappa_k S_t^k \mathrm{d}t \mid S_0, U_0\right] = \sum_{k=0}^{m} S_0^k A_k(U_0). \tag{3.11}$$

***Proof*:** Using the $\mathcal{F}_t$-martingale (3.6) provided in Theorem 3.1 and the martingale property, we have the expectation conditional on S_0 and U_0 at time $t = 0$ by

$$\mathbb{E}\left[Z_t + e^{-\theta t} \sum_{k=0}^{m} S_t^k A_k(U_t) \mid S_0, U_0\right] = \sum_{k=0}^{m} S_0^k A_k(U_0). \tag{3.12}$$

Setting $t = \infty$ in (3.12), (3.11) follows immediately. □

Applications of the Feynmann–Kac formula in general can be found in Karatzas & Shreve (1991). Its applications to financial mathematics can be noticed in Linetsky (1997, 2004, 2007) and the refereed papers therein. More recently, Goovaerts *et al.* (2012) constructed a recursive scheme for the Laplace transform of the transition density function of a diffusion process using the Feynmann–Kac formula, also see Shang *et al.* (2011).

4. Moments

In this section, we first derive Laplace transforms of the conditional moments and asymptotic moments of renewal shot-noise processes and discounted aggregate claims, respectively. Then, by inverting the Laplace transforms, we obtain the asymptotic moments and the first conditional moments in explicit forms. They are the main contribution of this paper. As examples, the associated first two moments and variances are discussed in more details.

4.1. Moments of renewal shot-noise processes

Theorem 4.1: *The Laplace transform (with respect to time t) of the mth moment of S_t conditional on S_0 and U_0 is given by*

$$\hat{e}_m(\theta; S_0, U_0; \delta) = \sum_{k=0}^{m} S_0^k A_k^*(U_0), \tag{4.1}$$

where the series of functions $\{A_k^*(u)\}_{k=0,1,\ldots,m}$ *can be iteratively solved from the system of equations*

$$A_m^*(u) := \frac{1}{\theta + m\delta}, \tag{4.2}$$

$$A_k^*(u) := \frac{\sum_{n=k+1}^m A_n^*(0)\binom{n}{k}\mu_{n-k}}{1-\hat{p}(\theta+\delta k)} \frac{e^{(\theta+\delta k)u}}{\bar{P}(u)} \int_u^\infty e^{-(\theta+\delta k)v} p(v)\mathrm{d}v, \quad k = m-1, m-2, \ldots, 1, 0, \tag{4.3}$$

with

$$A_m^*(0) = \frac{1}{\theta + m\delta},$$

$$A_k^*(0) = \frac{\hat{p}(\theta+\delta k)}{1-\hat{p}(\theta+\delta k)} \sum_{n=k+1}^m A_n^*(0)\binom{n}{k}\mu_{n-k}, \qquad k = m-1, m-2, \ldots, 1, 0.$$

Proof: Firstly, we express (3.11) in terms of Laplace transforms by

$$\sum_{k=1}^m \kappa_k \hat{e}_k(\theta; S_0, U_0; \delta) = \sum_{k=0}^m S_0^k A_k(U_0). \tag{4.4}$$

Setting $\kappa_m = 1$ and $\kappa_k = 0$ for all $k = 1, \ldots, m-1$ in (4.4), (3.7) and (3.8), we have (4.1), (4.2) and (4.3), respectively. Further setting $u = 0$ in (4.3), we have

$$A_k^*(0) = \frac{\sum_{n=k+1}^m A_n^*(0)\binom{n}{k}\mu_{n-k}}{1-\hat{p}(\theta+\delta k)} \int_0^\infty e^{-(\theta+\delta k)v} p(v)\mathrm{d}v = \frac{\hat{p}(\theta+\delta k)}{1-\hat{p}(\theta+\delta k)} \sum_{n=k+1}^m A_n^*(0)\binom{n}{k}\mu_{n-k}.$$

□

Based on Theorem 4.1, it is straightforward to obtain the Laplace transform for any conditional moment of S_t. For example, the Laplace transforms of the first two moments are specified as below.

Corollary 4.1: *The Laplace transform of the first moment of* S_t *conditional on* S_0 *and* U_0 *is given by*

$$\hat{e}_1(\theta; S_0, U_0; \delta) = \frac{S_0}{\theta+\delta} + \frac{\mu_1}{1-\hat{p}(\theta)} \frac{1}{\theta+\delta} \frac{e^{\theta U_0}}{\bar{P}(U_0)} \int_{U_0}^\infty e^{-\theta v} p(v)\mathrm{d}v. \tag{4.5}$$

Proof: Set $\kappa_2 = 0$ and $\kappa_1 = 1$ in (3.7) and (3.8), then, we have

$$A_2^*(u) = 0, \qquad A_1^*(u) = \frac{1}{\theta+\delta}, \qquad A_0^*(u) = \frac{\mu_1}{1-\hat{p}(\theta)} \frac{1}{\theta+\delta} \frac{e^{\theta u}}{\bar{P}(u)} \int_u^\infty e^{-\theta v} p(v)\mathrm{d}v.$$

From (4.4), the result follows. □

Corollary 4.2: *The Laplace transform of the second moment of* S_t *conditional on* S_0 *and* U_0 *is given by*

$$\hat{e}_2\left(\theta; S_0, U_0; \delta\right) = \frac{S_0^2}{\theta+2\delta} + \frac{S_0}{\theta+2\delta} \frac{2\mu_1}{1-\hat{p}(\theta+\delta)} \frac{e^{(\theta+\delta)U_0}}{\bar{P}(U_0)} \int_{U_0}^\infty e^{-(\theta+\delta)v} p(v)\mathrm{d}v$$
$$+ \frac{\frac{\mu_2}{\theta+2\delta} + \frac{1}{\theta+2\delta} \frac{2\mu_1^2 \hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)}}{1-\hat{p}(\theta)} \frac{e^{\theta U_0}}{\bar{P}(U_0)} \int_{U_0}^\infty e^{-\theta v} p(v)\mathrm{d}v. \tag{4.6}$$

***Proof*:** Setting $\kappa_2 = 1$ and $\kappa_1 = 0$ in (3.7) and (3.8), we have

$$\begin{aligned} A_2^*(u) &= \frac{1}{\theta + 2\delta}, \\ A_1^*(u) &= \frac{1}{\theta + 2\delta} \frac{2\mu_1}{1 - \hat{p}(\theta + \delta)} \frac{e^{(\theta+\delta)u}}{\bar{P}(u)} \int_u^\infty e^{-(\theta+\delta)v} p(v) \mathrm{d}v, \\ A_0^*(u) &= \frac{\frac{\mu_2}{\theta+2\delta} + \frac{1}{\theta+2\delta} \frac{2\mu_1^2 \hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)}}{1 - \hat{p}(\theta)} \frac{e^{\theta u}}{\bar{P}(u)} \int_u^\infty e^{-\theta v} p(v) \mathrm{d}v. \end{aligned}$$

From (4.4), the result follows. □

The distribution converges pretty fast, and we can easily observe how the distribution converges to its asymptotic distribution via its mean.

Corollary 4.3: *We have the asymptotics of the first moment,*

$$e_1(t; S_0, U_0; \delta) = d_0 + d_1 e^{-\delta t} + o\left(e^{-\delta t}\right), \tag{4.7}$$

where

$$d_0 := \frac{\mu_1}{\delta \gamma_1}, \qquad d_1 := S_0 + \frac{\mu_1}{\bar{P}(U_0)} \frac{e^{-\delta U_0}}{1 - \hat{p}(-\delta)} \int_{U_0}^\infty e^{\delta v} p(v) \mathrm{d}v.$$

***Proof*:** The Laplace transform of the first moment of S_t conditional on S_0 and U_0 is given by (4.5). We know that the limit $\lim_{t\to\infty} e_1(t; S_0, U_0; \delta)$ exists, more precisely,

$$\begin{aligned} d_0 &:= \lim_{t\to\infty} e_1(t; S_0, U_0; \delta) \\ &= \lim_{\theta\to 0} \theta \hat{e}_1(\theta; S_0, U_0; \delta) \\ &= \frac{\mu_1}{\bar{P}(U_0)} \int_{U_0}^\infty \lim_{\theta\to 0} \left[\frac{1}{\theta + \delta} \frac{\theta}{1 - \hat{p}(\theta)} e^{\theta(U_0 - v)} \right] p(v) \mathrm{d}v \\ &= \frac{1}{\delta} \frac{1}{\gamma_1} \frac{\mu_1}{\bar{P}(U_0)} \int_{U_0}^\infty p(v) \mathrm{d}v \\ &= \frac{\mu_1}{\delta \gamma_1}. \end{aligned}$$

Define the function

$$g(t) := e^{\delta t} \Big[e_1(t; S_0, U_0; \delta) - d_0 \Big],$$

and its Laplace transform

$$\hat{g}(\theta) := \int_0^\infty e^{-\theta t} g(t) \mathrm{d}t.$$

Then, we have

$$\begin{aligned} d_1 &:= \lim_{t\to\infty} g(t) \\ &= \lim_{\theta\to 0} \theta\hat{g}(\theta) \\ &= \lim_{\theta\to 0} \theta\left[\hat{e}_1\big(\theta-\delta; S_0, U_0; \delta\big) - \frac{d_0}{\theta-\delta}\right] \\ &= \lim_{\theta\to 0} \theta\left[\frac{1}{\theta}S_0 + \frac{1}{\theta}\frac{\mu_1}{1-\hat{p}(\theta-\delta)}\frac{e^{(\theta-\delta)U_0}}{\bar{P}(U_0)}\int_{U_0}^{\infty} e^{-(\theta-\delta)v}p(v)\mathrm{d}v\right] \\ &= S_0 + \frac{\mu_1}{\bar{P}(U_0)}\int_{U_0}^{\infty}\lim_{\theta\to 0}\left[\frac{e^{(\theta-\delta)(U_0-v)}}{1-\hat{p}(\theta-\delta)}\right]p(v)\mathrm{d}v \\ &= S_0 + \frac{\mu_1}{\bar{P}(U_0)}\frac{e^{-\delta U_0}}{1-\hat{p}(-\delta)}\int_{U_0}^{\infty} e^{\delta v}p(v)\mathrm{d}v. \end{aligned}$$

Therefore, we have (4.7). □

We can see from (4.7) that the conditional moment converges at an exponential rate with respect to time t, and the asymptotic results could provide reasonable approximations to their moments and distributions.

The initial value U_0 is usually unknown in practice. To calculate the first two conditional moments of S_t for actuarial applications, we assign the asymptotic (or limiting) distribution of U_t to U_0 for mathematical convenience, which can provide reasonable approximations and also substantially simplify the expressions of the Laplace transforms of moments. To do so, we first state a proposition in Cox & Miller (1965, p. 347) or Cox (1962, p. 61), which is a well-known result in renewal theory.

Proposition 4.1: *The asymptotic (or limiting) distribution of U_t, denoted by Π, has the density function*

$$f_{\Pi}(u) := \frac{\bar{P}(u)}{\gamma_1} = \frac{1}{\gamma_1}\exp\left(-\int_0^u \rho(v)\mathrm{d}v\right), \qquad u \geq 0.$$

Π is in fact the limiting distribution of U_t when $t \to \infty$, and it can serve a reasonable approximation for the distribution of U_t when the underlying process has been running for a relatively long period and is close to the stationary (asymptotic) state (Cox 1962, Chapter 5, p. 61–70).

Now, let us start with finding the asymptotic mth moment of S_t when $U_0 \sim \Pi$, denoted by

$$e_m(t; S_0; \delta) := \mathbb{E}\big[S_t^m \mid S_0\big].$$

Denote the Laplace transform (with respect to time t) of the mth moment of S_t conditional on S_0 by

$$\hat{e}_m(\theta; S_0; \delta) := \mathbb{E}\big[\hat{e}_m(\theta; S_0, U_0)\big], \qquad U_0 \sim \Pi.$$

Proposition 4.2: *If $U_0 \sim \Pi$, then, we have*

$$\mathbb{E}\left[\frac{e^{\xi U_0}}{\bar{P}(U_0)}\int_{U_0}^{\infty} e^{-\xi v}p(v)\mathrm{d}v\right] = \frac{1}{\gamma_1}\frac{1-\hat{p}(\xi)}{\xi}.$$

***Proof*:**

$$
\begin{aligned}
\mathbb{E}\left[\frac{e^{\xi U_0}}{\bar{P}(U_0)}\int_{U_0}^{\infty} e^{-\xi v}p(v)\mathrm{d}v\right] &= \int_{u=0}^{\infty}\frac{e^{\xi u}}{\bar{P}(u)}\int_{v=u}^{\infty} e^{-\xi v}p(v)\mathrm{d}v f_{\Pi}(u)\mathrm{d}u \\
&= \int_{u=0}^{\infty}\int_{v=u}^{\infty} e^{-\xi v}p(v)\frac{e^{\xi u}}{\gamma_1}\mathrm{d}v\mathrm{d}u \\
&= \int_{v=0}^{\infty}\int_{u=0}^{v} e^{-\xi v}p(v)\frac{e^{\xi u}}{\gamma_1}\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{\gamma_1}\int_{v=0}^{\infty} e^{-\xi v}p(v)\left(\int_{u=0}^{v} e^{\xi u}\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{\gamma_1}\int_{v=0}^{\infty} e^{-\xi v}p(v)\frac{e^{\xi v}-1}{\xi}\mathrm{d}v \\
&= \frac{1}{\gamma_1}\int_{v=0}^{\infty} p(v)\frac{1-e^{-\xi v}}{\xi}\mathrm{d}v \\
&= \frac{1}{\gamma_1}\frac{1-\hat{p}(\xi)}{\xi}.
\end{aligned}
$$

□

Theorem 4.2: *For $U_0 \sim \Pi$, the Laplace transform of the mth moment of S_t conditional on S_0 is given by*

$$
\hat{e}_m(\theta; S_0; \delta) = \sum_{k=0}^{m} B_k(\theta)S_0^k, \tag{4.8}
$$

where

$$
\begin{aligned}
B_m(\theta) &:= \frac{1}{\theta+m\delta}, \\
B_k(\theta) &:= \frac{1}{\gamma_1}\frac{1}{\theta+\delta k}\sum_{n=k+1}^{m} A_n^*(0)\binom{n}{k}\mu_{n-k}, \qquad k = m-1, m-2, \ldots, 1, 0.
\end{aligned} \tag{4.9}
$$

***Proof*:** Using (4.1) and Proposition 4.1, we have

$$
\hat{e}_m(\theta; S_0; \delta) = \mathbb{E}\big[\hat{e}_m(\theta; S_0, U_0; \delta)\big] = \mathbb{E}\left[\sum_{k=0}^{m} S_0^k A_k^*(U_0)\right] = \sum_{k=0}^{m} S_0^k \mathbb{E}\big[A_k^*(U_0)\big],
$$

where

$$
\begin{aligned}
B_k(\theta) := \mathbb{E}\left[A_k^*(U_0)\right] &= \frac{\sum_{n=k+1}^{m} A_n^*(0)\binom{n}{k}\mu_{n-k}}{1-\hat{p}(\theta+\delta k)}\mathbb{E}\left[\frac{e^{(\theta+\delta k)U_0}}{\bar{P}(U_0)}\int_{U_0}^{\infty} e^{-(\theta+\delta k)v}p(v)\mathrm{d}v\right] \\
&= \frac{\sum_{n=k+1}^{m} A_n^*(0)\binom{n}{k}\mu_{n-k}}{1-\hat{p}(\theta+\delta k)}\frac{1}{\gamma_1}\frac{1-\hat{p}(\theta+\delta k)}{\theta+\delta k} \\
&= \frac{1}{\gamma_1}\frac{1}{\theta+\delta k}\sum_{n=k+1}^{m} A_n^*(0)\binom{n}{k}\mu_{n-k}, \qquad k = m-1, m-2, \ldots, 1, 0.
\end{aligned}
$$

□

Theorem 4.3: *For $U_0 \sim \Pi$, the mth asymptotic moment of S_t is given by*

$$\lim_{t\to\infty} \mathbb{E}\left[S_t^m \mid S_0\right] = \frac{1}{\gamma_1}\sum_{k=1}^{m} \mu_k A_k^{**}, \tag{4.10}$$

where the constants $\{A_k^{**}\}_{k=1,\dots,m}$ can be calculated iteratively from

$$A_m^{**} := \frac{1}{m\delta},$$
$$A_k^{**} := \frac{\hat{p}(\delta k)}{1-\hat{p}(\delta k)} \sum_{n=k+1}^{m} A_n^{**}\binom{n}{k}\mu_{n-k}, \qquad k = m-1, m-2, \dots, 1.$$

***Proof*:** By the *final value theorem*, we have

$$\lim_{t\to\infty} \mathbb{E}\left[S_t^m \mid S_0\right] = \lim_{\theta\to 0} \theta \hat{e}_m(\theta; S_0) = \sum_{k=0}^{m} S_0^k \lim_{\theta\to 0} \theta B_k(\theta) = \lim_{\theta\to 0} \theta B_0(\theta),$$

since

$$\lim_{\theta\to 0} \theta B_m(\theta) = 0,$$
$$\lim_{\theta\to 0} \theta B_k(\theta) = \frac{1}{\gamma_1} \sum_{n=k+1}^{m} \binom{n}{k}\mu_{n-k} \lim_{\theta\to 0} \frac{\theta}{\theta+\delta k} A_n^*(0) = 0, \qquad k = m-1, m-2, \dots, 1.$$

Note that, according to (4.9), we have

$$B_0(\theta) = \frac{1}{\gamma_1}\frac{1}{\theta}\sum_{n=1}^{m} A_n^*(0)\mu_n,$$

then,

$$\lim_{\theta\to 0} \theta B_0(\theta) = \frac{1}{\gamma_1}\sum_{n=1}^{m} \mu_n \lim_{\theta\to 0} A_n^*(0) = \frac{1}{\gamma_1}\sum_{k=1}^{m} \mu_k \lim_{\theta\to 0} A_k^*(0) = \frac{1}{\gamma_1}\sum_{k=1}^{m} \mu_k A_k^{**},$$

where $A_k^{**} = \lim_{\theta\to 0} A_k^*(0)$. □

Now, let us start with finding the first moment of S_t conditional on S_0 at time $t = 0$ by inverting its Laplace transform.

Corollary 4.4: *For $U_0 \sim \Pi$, the first moment of S_t conditional on S_0 is given by*

$$e_1(t; S_0; \delta) = S_0 e^{-\delta t} + \frac{\mu_1}{\gamma_1}\left(\frac{1-e^{-\delta t}}{\delta}\right), \tag{4.11}$$

and the first asymptotic moment is given by

$$\lim_{t\to\infty} \mathbb{E}[S_t \mid S_0] = \frac{\mu_1}{\gamma_1 \delta}. \tag{4.12}$$

Proof: Using (4.6) and Proposition 4.1, and setting $\xi = \theta$ in Proposition 4.2, the Laplace transform (with respect to time t) of the first moment of S_t conditional on S_0 for $U_0 \sim \Pi$ is given by

$$\begin{aligned}\hat{e}_1(\theta; S_0; \delta) &= \frac{S_0}{\theta+\delta} + \frac{\mu_1}{1-\hat{p}(\theta)}\frac{1}{\theta+\delta}\mathbb{E}\left[\frac{e^{\theta U_0}}{\bar{P}(U_0)}\int_{U_0}^{\infty} e^{-\theta v}p(v)\mathrm{d}v\right] \\ &= \frac{S_0}{\theta+\delta} + \frac{\mu_1}{\gamma_1}\frac{1}{\theta(\theta+\delta)} \\ &= S_0\frac{1}{\theta+\delta} + \frac{\mu_1}{\gamma_1}\frac{1}{\delta}\left(\frac{1}{\theta} - \frac{1}{\theta+\delta}\right).\end{aligned}$$

Inverting it immediately gives us (4.11). Note that, we have Laplace transforms

$$\mathcal{L}_\theta\left\{e^{-\delta t}\right\} = \frac{1}{\theta+\delta}, \qquad \mathcal{L}_\theta\left\{1\right\} = \frac{1}{\theta}.$$

Its asymptotic result in (4.12) follows by setting $t \to \infty$ in (4.11). □

Unfortunately, in general, it is not possible for us to obtain other conditional moments explicitly beyond the first moments. Therefore, we have to develop numerical methods for estimation.

Corollary 4.5: *For $U_0 \sim \Pi$, the second moment of S_t conditional on S_0 is given by*

$$e_2(t; S_0; \delta) = S_0^2 e^{-2\delta t} + \frac{2\mu_1}{\delta\gamma_1}S_0\left(e^{-\delta t} - e^{-2\delta t}\right) + \frac{\mu_2}{\gamma_1}\left(\frac{1-e^{-2\delta t}}{2\delta}\right) + \frac{\mu_1^2}{\delta\gamma_1}\frac{\hat{p}(\delta)}{1-\hat{p}(\delta)}F_4(t), \tag{4.13}$$

where $F_4(t)$ is a function of time t with the Laplace transform

$$\hat{F}_4(\theta) = \frac{1-\hat{p}(\delta)}{\hat{p}(\delta)}\frac{2\delta}{\theta(\theta+2\delta)}\frac{\hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)}; \tag{4.14}$$

The asymptotic second moment is given by

$$\frac{1}{\gamma_1\delta}\left[\frac{\mu_2}{2} + \mu_1^2\frac{\hat{p}(\delta)}{1-\hat{p}(\delta)}\right]. \tag{4.15}$$

Proof: Using (4.5) and Propositions 4.1 and 4.2, the Laplace transform (with respect to time t) of the second moment of S_t conditional on S_0 for $U_0 \sim \Pi$ is given by

$$\begin{aligned}&\hat{e}_2(\theta; S_0; \delta) \\ &= \frac{S_0^2}{\theta+2\delta} + \frac{2\mu_1 S_0}{(\theta+2\delta)(\theta+\delta)}\frac{1}{\gamma_1} + \left[\frac{\mu_2}{\theta+2\delta} + \frac{\hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)}\frac{2\mu_1^2}{\theta+2\delta}\right]\frac{1}{\theta}\frac{1}{\gamma_1} \\ &= S_0^2\frac{1}{\theta+2\delta} + \frac{2\mu_1 S_0}{\gamma_1}\frac{1}{\delta}\left(\frac{1}{\theta+\delta} - \frac{1}{\theta+2\delta}\right) + \frac{\mu_2}{\gamma_1}\frac{1}{2\delta}\left(\frac{1}{\theta} - \frac{1}{\theta+2\delta}\right) \\ &\quad + \frac{2\mu_1^2}{\gamma_1}\frac{1}{\theta}\frac{\hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)}\frac{1}{\theta+2\delta} \qquad (4.16) \\ &= S_0^2\frac{1}{\theta+2\delta} + \frac{2\mu_1 S_0}{\gamma_1}\frac{1}{\delta}\left(\frac{1}{\theta+\delta} - \frac{1}{\theta+2\delta}\right) + \frac{\mu_2}{\gamma_1}\frac{1}{2\delta}\left(\frac{1}{\theta} - \frac{1}{\theta+2\delta}\right) \\ &\quad + \frac{\mu_1^2}{\delta\gamma_1}\frac{\hat{p}(\delta)}{1-\hat{p}(\delta)}\hat{F}_4(\theta). \qquad (4.17)\end{aligned}$$

The first three terms of (4.17) can be inverted analytically, then, we obtain (4.13). Based on Theorem 4.10, setting $m = 2$, we can calculate

$$A_2^{**} = \frac{1}{2\delta}, \qquad A_1^{**} = \frac{\hat{p}(\delta)}{1-\hat{p}(\delta)}\frac{\mu_1}{\delta}.$$

Substituting A_1^{**} and A_2^{**} into (4.10), we derive (4.15). □

Corollary 6.3 in Léveillé & Garrido (2001a, p. 230) confirms both our results (4.12) and (4.15). Interestingly, from a probabilistic point of view, function $F_4(t)$ in (4.14) can be nicely interpreted as the CDF of a random time τ^*, which can be estimated by the following algorithm:

Algorithm 4.1 (Decomposition Approach): *The random time τ^* can be exactly sampled by the following distributional decomposition:*

$$\tau^* \stackrel{\mathcal{D}}{=} E^* + \sum_{i=1}^{I} E_i, \tag{4.18}$$

where

- *E^* is an exponential random variable of constant rate 2δ, i.e. $E^* \sim \mathrm{Exp}(2\delta)$;*
- *I is a geometric random variable with success probability parameter $1-\hat{p}(\delta)$, i.e. $I \sim \mathrm{Geometric}\big(1-\hat{p}(\delta)\big)$ with the probability mass distribution*

$$\Pr\{I = i\} = \hat{p}^{i-1}(\delta)\big[1-\hat{p}(\delta)\big], \qquad i = 1, 2, 3, \ldots;$$

- *$\{E_i\}_{i=1,2,\ldots}$ are i.i.d. random variables with the identical Laplace transform*

$$\hat{f}_{E_i}(\theta) := \mathbb{E}\left[e^{-\theta E_i}\right] = \frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)}. \tag{4.19}$$

Proof: The Laplace transform of $F_4(t)$ specified by (4.14) can be rewritten as

$$\begin{aligned}
\hat{F}_4(\theta) &= \frac{1}{\theta} \times 2\delta \frac{1-\hat{p}(\delta)}{\hat{p}(\delta)} \frac{\hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)} \frac{1}{\theta+2\delta} \\
&= \frac{1}{\theta} \times 2\delta \frac{1-\hat{p}(\delta)}{\hat{p}(\delta)} \frac{1}{\theta+2\delta} \left[\frac{1}{1-\hat{p}(\theta+\delta)} - 1\right] \\
&= \frac{1}{\theta} \times 2\delta \frac{1-\hat{p}(\delta)}{\hat{p}(\delta)} \frac{1}{\theta+2\delta} \sum_{i=1}^{\infty} \hat{p}^i(\theta+\delta) \\
&= \frac{1}{\theta} \times \sum_{i=1}^{\infty} \hat{p}^{i-1}(\delta)\big[1-\hat{p}(\delta)\big] \left[\frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)}\right]^i \frac{2\delta}{\theta+2\delta} \\
&= \frac{1}{\theta} \times \sum_{i=1}^{\infty} \Pr\{I = i\} \left[\frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)}\right]^i \frac{2\delta}{\theta+2\delta} \\
&= \frac{1}{\theta} \times \mathbb{E}\left[\left[\frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)}\right]^I\right] \frac{2\delta}{\theta+2\delta} \\
&= \frac{1}{\theta} \times \int_0^{\infty} e^{-\theta t} f_4(t) \mathrm{d}t,
\end{aligned}$$

where $f_4(t)$ can be considered as the density function of the random time τ^* defined by (4.18), and the associated CDF is given by

$$F_4(t) := \Pr\left\{\tau^* \le t\right\} = \mathbb{E}\left[\mathbf{1}\{\tau^* \le t\}\right].$$

The Laplace transform of the CDF is given by

$$\hat{F}_4(\theta) := \int_0^\infty e^{-\theta t} F_4(t)\mathrm{d}t = \frac{1}{\theta}\hat{f}_4(\theta),$$

where

$$\begin{aligned}
\hat{f}_4(\theta) &:= \int_0^\infty e^{-\theta t} f_4(t)\mathrm{d}t \\
&= \frac{1-\hat{p}(\delta)}{\hat{p}(\delta)} \frac{\hat{p}(\theta+\delta)}{1-\hat{p}(\theta+\delta)} \frac{2\delta}{\theta+2\delta} \\
&= \mathbb{E}\left[\left[\frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)}\right]^I\right] \times \frac{2\delta}{\theta+2\delta} \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-\theta\sum_{i=1}^I E_i}\right]\right] \times \mathbb{E}\left[e^{-\theta E^*}\right] \\
&= \mathbb{E}\left[e^{-\theta\left(E^*+\sum_{i=1}^I E_i\right)}\right] \\
&= \mathbb{E}\left[e^{-\theta\tau^*}\right].
\end{aligned} \tag{4.20}$$

So, we have the decomposition (4.18). Note that, $E_1, E_2, \ldots, E_I$ have the identical Laplace transform (4.19), and they are well defined random variables, since

$$\hat{f}_{E_i}(\theta) = \frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)} = \int_0^\infty e^{-\theta t} \frac{e^{-\delta t}}{\hat{p}(\delta)} p(t)\mathrm{d}t = \int_0^\infty e^{-\theta t} f_{E_i}(t)\mathrm{d}t,$$

and we have the density function of E_i via the *Esscher transform* (Gerber & Shiu 1994) (or, *exponential tilting*) as

$$f_{E_i}(t) = \frac{e^{-\delta t}}{\hat{p}(\delta)} p(t), \tag{4.21}$$

and

$$\int_0^\infty f_{E_i}(t)\mathrm{d}t = \int_0^\infty \frac{e^{-\delta t}}{\hat{p}(\delta)} p(t)\mathrm{d}t = \frac{\hat{p}(\delta)}{\hat{p}(\delta)} = 1.$$

□

Note that, since $F_4(t)$ can be interpreted as a CDF, we have

$$\lim_{t\to\infty} F_4(t) = 1.$$

Setting $t \to \infty$ in (4.13), again, we obtain the asymptotic result (4.15). Alternatively, $F_4(t)$ as a CDF could be estimated by the numerical inversion of Laplace transform (Abate & Whitt 1992, 1995, 2006), which will be discussed in detail in Section 5.

Corollary 4.6: *For $U_0 \sim \Pi$, the variance of S_t conditional on S_0 is given by*

$$\mathrm{Var}\left[S_t \mid S_0\right] = \frac{\mu_2}{\gamma_1}\left(\frac{1-e^{-2\delta t}}{2\delta}\right) - \frac{\mu_1^2}{\gamma_1^2}\left(\frac{1-e^{-\delta t}}{\delta}\right)^2 + \frac{\mu_1^2}{\delta\gamma_1}\frac{\hat{p}(\delta)}{1-\hat{p}(\delta)} F_4(t). \tag{4.22}$$

***Proof*:** Based on the first moment (4.11) and the second moment (4.13), we have the variance

$$\mathrm{Var}\left[S_t \mid S_0\right] = \mathbb{E}\left[S_t^2 \mid S_0\right] - \left(\mathbb{E}\left[S_t \mid S_0\right]\right)^2.$$

□

The moments, of course, can be estimated by the *direct simulation* for sample paths of S_t: Say, to estimate the moments of S_t at time $T > 0$, we have to simulate all interarrival times, jump sizes within the time period $[0, T]$; and moreover, as intermediate steps required, we also need to solve all ODEs recursively between two successive jumps, in order exactly simulate the distribution of S_t at an arbitrary time point T. In fact, it is a *path-dependent approach*. However, our *decomposition approach* provides a shortcut, which avoids simulating full paths of the underlying stochastic processes but only needs a few simple random variables as illustrated in Algorithm 4.1. Essentially, we use a Monte Carlo alternative Laplace transform inversion.

4.2. Moments of discounted aggregate claims

Denote the mth moment of L_t^0 by

$$\ell_m(t) := \mathbb{E}\left[\left(L_t^0\right)^m\right],$$

and the associated Laplace transform by

$$\hat{\ell}_m(\theta) := \int_0^\infty e^{-\theta t}\ell_m(t)\mathrm{d}t,$$

which can be obtained explicitly as below.

Theorem 4.4: *For $U_0 \sim \Pi$, the Laplace transform of the mth moment of L_t^0 is given by*

$$\hat{\ell}_m(\theta) = \frac{1}{\gamma_1}\frac{1}{\theta + mr}\sum_{n=1}^{m}\mu_n\tilde{A}_n, \tag{4.23}$$

where

$$\tilde{A}_m = \frac{1}{\theta},$$

$$\tilde{A}_k = \frac{\hat{p}\big(\theta + (m-k)r\big)}{1 - \hat{p}\big(\theta + (m-k)r\big)}\sum_{n=k+1}^{m}\tilde{A}_n\binom{n}{k}\mu_{n-k}, \qquad k = m-1, m-2, \ldots, 1, 0.$$

***Proof*:** Note that, by setting

$$S_0 = 0, \qquad L_t^0 = e^{-rt}S_t, \qquad \delta = -r,$$

in the process S_t, we recover the associated L_t^0. Using this duality property, in general, we have the mth moment of L_t^0 by

$$\mathbb{E}\left[\left(L_t^0\right)^m\right] = e^{-mrt}e_m(t; 0; -r).$$

Then, we have its Laplace transform

$$\hat{\ell}_m(\theta) = \mathcal{L}_\theta\left\{e^{-mrt}e_m(t; 0; -r)\right\} = \hat{e}_m(\theta + mr; 0; -r),$$

where $\hat{e}_m(\theta; S_0; \delta)$ is specified by (4.8). □

Based on the Laplace transform (4.23), as examples, we can compute the first two moments and the variance as below.

Corollary 4.7: *The first moment and the variance of L_t^0 are given by*

$$\ell_1(t) = \frac{\mu_1}{\gamma_1}\left(\frac{1-e^{-rt}}{r}\right), \tag{4.24}$$

$$\ell_2(t) = \frac{\mu_2}{\gamma_1}\left(\frac{1-e^{-2rt}}{2r}\right) + \frac{\mu_1^2}{r\gamma_1}\frac{\hat{p}(r)}{1-\hat{p}(r)}\tilde{F}_4(t), \tag{4.25}$$

$$\mathrm{Var}\left[L_t^0\right] = \frac{\mu_2}{\gamma_1}\left(\frac{1-e^{-2rt}}{2r}\right) - \frac{\mu_1^2}{\gamma_1^2}\left(\frac{1-e^{-rt}}{r}\right)^2 + \frac{\mu_1^2}{r\gamma_1}\frac{\hat{p}(r)}{1-\hat{p}(r)}\tilde{F}_4(t), \tag{4.26}$$

where $\tilde{F}_4(t)$ is the CDF of random time $\tilde{\tau}^*$ which can be exactly simulated the same as τ^* via Algorithm 4.1 *by replacing δ by r.*

Proof: Setting

$$S_0 = 0, \qquad L_t^0 = e^{-rt}S_t, \qquad \delta = -r,$$

in (4.11), we have the mean (4.24), i.e.

$$\ell_1(t) = \mathbb{E}\left[L_t^0\right] = e^{-rt}e_1(t;0;-r).$$

Similarly, for the second moment, based on (4.13), we have

$$\ell_2(t) = \mathbb{E}\left[\left(L_t^0\right)^2\right] = e^{-2rt}e_2(t;0;-r),$$

with its Laplace transform

$$\mathcal{L}_\theta\left\{\ell_2(t)\right\} = \mathcal{L}_\theta\left\{e^{-2rt}e_2(t;0;-r)\right\} = \hat{e}_2(\theta+2r;0;-r),$$

where $\hat{e}_2(\theta;S_0;\delta)$ is specified by (4.16). Then, we have the Laplace transform

$$\mathcal{L}_\theta\left\{\ell_2(t)\right\} = \frac{\mu_2}{\gamma_1}\frac{1}{2r}\left(\frac{1}{\theta} - \frac{1}{\theta+2r}\right) + \frac{2\mu_1^2}{\gamma_1}\frac{1}{\theta+2r}\frac{\hat{p}(\theta+r)}{1-\hat{p}(\theta+r)}\frac{1}{\theta},$$

which is exactly the same as the last two terms of (4.16) by replacing δ by r. Therefore, we have the second moment (4.26). Finally, it is straightforward to obtain the variance (4.26). □

5. Numerical illustration with applications

To illustrate the applicability of renewal shot-noise processes and our newly-derived results, in this section, we offer four choices for modelling renewal interarrival times: (1) exponential (Exp), (2) gamma, (3) inverse Gaussian (IG) and (4) folded normal (FN) distributions. The first two examples are for actuarial application of discounted aggregate claims. Since the discounted aggregate claims L_t^0 defined by (2.1) can be alternatively interpreted as the present value of aggregate losses from a portfolio in general, we use the third and fourth examples for credit risk and reliability applications, respectively. We commonly assume $S_0 = 1$ and $\delta = r = 0.05$, and the jump sizes follow an exponential distribution of unit rate, i.e. $\mu_1 = 1, \mu_2 = 2$ for all four cases.

For each case, we compute the first conditional moments and variances of renewal shot-noise process and discounted aggregate claims, respectively. Except for the first case of exponential distribution, it is often not easy to obtain explicit expression for $F_4(t)$ in (4.22) and $\tilde{F}_4(t)$ in (4.26).

We have to rely on estimating $F_4(t)$ by Monte Carlo *exact simulation* (ES)[1] via Algorithm 4.1, or, numerical inversion (NI) of Laplace transform such as *Euler algorithm* and *Talbot algorithm* (Abate & Whitt 2006, p. 415–416). The detailed implementation of NI we adopted in this paper is explained in Appendix 1. However, it is well known that the algorithms for numerical inversion are not perfect, and they are often not reliable when the underlying function has some discontinuity or oscillation, or the function of Laplace transform involves complex special functions. Therefore, efficient simulation becomes a crucial and more reliable alternative tool for estimation. Based on the fact that the shape of true function $F_4(t)$ or $\tilde{F}_4(t)$ beyond the exponential case is unknown, it is prudent for us to implement the two estimation approaches of ES and NI simultaneously in order to validate each other.

5.1. Example: Poisson shot-noise process

If N_t is a Poisson process, i.e. the interarrival times follow a simple exponential distribution, then, S_t is the classical Poisson shot-noise process and explicit results for variances exist. This special case was investigated by Jang (2004), and same results can be recovered here. In fact, this provides a benchmark case that can be used for validating the estimation methods ES and NI for computing $F_4(t)$ and $\tilde{F}_4(t)$ in the conditional second moments and variances. If $\tau_i \sim \text{Exp}(\varrho)$, $\varrho > 0$, with the density function

$$p(u) = \varrho e^{-\varrho u},$$

we have

$$\hat{p}(\theta) = \frac{\varrho}{\varrho + \theta}, \qquad \gamma_1 = \frac{1}{\varrho}.$$

From (4.11), we have the first moment

$$\mathbb{E}\left[S_t \mid S_0\right] = S_0 e^{-\delta t} + \mu_1 \varrho \left(\frac{1 - e^{-\delta t}}{\delta}\right).$$

The first moment of discounted aggregate claims (i.e. the actuarial net premium) at present time 0 is given by

$$\mathbb{E}\left[L_t^0\right] = \mu_1 \varrho \left(\frac{1 - e^{-rt}}{r}\right),$$

which can be also found in Léveillé & Garrido (2001a, 2001b), Jang (2004) and Jang & Krvavych (2004). For calculating the associated variances, from (4.20), we have

$$\hat{f}_4(\theta) = 2\delta \left(\frac{1}{\theta + \delta} - \frac{1}{\theta + 2\delta}\right), \qquad \hat{F}_4(\theta) = \frac{2\delta}{\theta} \left(\frac{1}{\theta + \delta} - \frac{1}{\theta + 2\delta}\right),$$

which can be inverted analytically, respectively, i.e.

$$f_4(t) = 2\delta \left(e^{-\delta t} - e^{-2\delta t}\right), \qquad F_4(t) = \left(1 - e^{-\delta t}\right)^2.$$

Hence, based on (4.22) and (4.26), we have

$$\text{Var}\left[S_t \mid S_0\right] = \mu_2 \varrho \left(\frac{1 - e^{-2\delta t}}{2\delta}\right), \qquad \text{Var}\left[L_t^0\right] = \mu_2 \varrho \left(\frac{1 - e^{-2rt}}{2r}\right).$$

So, $\text{Var}\left[S_t \mid S_0\right]$ and $\text{Var}\left[L_t^0\right]$ are equal when $r = \delta$. We plot the conditional means, the true and estimated variances of S_t and L_t^0 for $\varrho = 1$ respectively in Figure 2, with numerical results reported in Table 1. Note that, each point in the variance plots is estimated by the exact simulation

[1] ES is a simulation method of drawing an unbiased associated estimator throughout the entire simulation process.

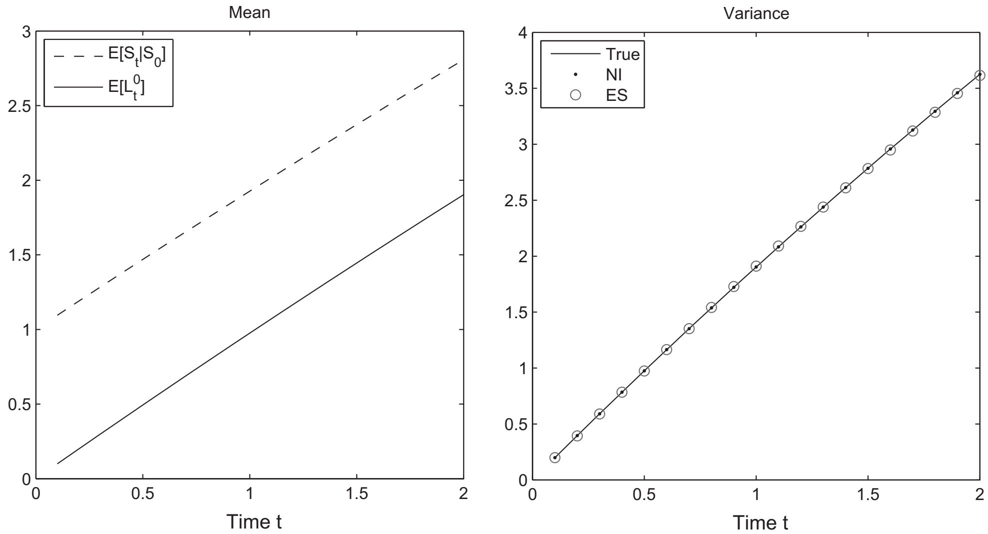

Figure 2. Means and variances of S_t and L_t^0 for the exponential (Exp) case with $\varrho = 1$; the associated detailed numerical results are reported in Table 1.

Table 1. Means and variances of S_t and L_t^0.

Time t	$\mathbb{E}[S_t \mid S_0]$	$\mathbb{E}[L_t^0]$	NI-Var	ES-Var	True-Var	$\mathbb{E}[S_t \mid S_0]$	$\mathbb{E}[L_t^0]$	NI-Var	ES-Var
			⟨Exp⟩				⟨Gamma⟩		
0.2	1.1891	0.1990	0.3960	0.3942	0.3960	1.1891	0.1990	0.3652	0.3655
0.4	1.3762	0.3960	0.7842	0.7843	0.7842	1.3762	0.3960	0.6860	0.6849
0.6	1.5615	0.5911	1.1647	1.1643	1.1647	1.5615	0.5911	0.9839	0.9790
0.8	1.7450	0.7842	1.5377	1.5411	1.5377	1.7450	0.7842	1.2685	1.2591
1.0	1.9266	0.9754	1.9033	1.9107	1.9033	1.9266	0.9754	1.5443	1.5322
1.2	2.1065	1.1647	2.2616	2.2668	2.2616	2.1065	1.1647	1.8131	1.8044
1.4	2.2845	1.3521	2.6128	2.6110	2.6128	2.2845	1.3521	2.0759	2.0649
1.6	2.4608	1.5377	2.9571	2.9485	2.9571	2.4608	1.5377	2.3333	2.3222
1.8	2.6353	1.7214	3.2946	3.2861	3.2946	2.6353	1.7214	2.5855	2.5695
2.0	2.8081	1.9033	3.6254	3.6152	3.6254	2.8081	1.9033	2.8325	2.8209
			⟨IG⟩				⟨FN⟩		
0.2	1.1891	0.1990	0.3621	0.3623		2.2371	1.2471	2.1387	2.1371
0.4	1.3762	0.3960	0.6992	0.7012		3.4619	2.4817	4.0478	4.0477
0.6	1.5615	0.5911	1.0390	1.0395		4.6745	3.7041	5.9246	5.8159
0.8	1.7450	0.7842	1.3805	1.3840		5.8751	4.9143	7.7669	7.6319
1.0	1.9266	0.9754	1.7215	1.7283		7.0637	6.1125	9.5747	9.6682
1.2	2.1065	1.1647	2.0605	2.0664		8.2405	7.2987	11.3487	11.5322
1.4	2.2845	1.3521	2.3965	2.4025		9.4056	8.4732	13.0894	13.4175
1.6	2.4608	1.5377	2.7288	2.7363		10.5591	9.6359	14.7973	14.8805
1.8	2.6353	1.7214	3.0568	3.0611		11.7011	10.7871	16.4730	16.5891
2.0	2.8081	1.9033	3.3802	3.3764		12.8317	11.9269	18.1169	18.3422

(ES) via Algorithm 4.1 with 10^7 replications. For implementing Algorithm 4.1, since an exponential distribution after the exponential tilting (4.21) is still an exponential distribution, i.e.

$$\hat{f}_{E_i}(\theta) = \frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)} = \frac{\frac{\varrho}{\varrho+\theta+\delta}}{\frac{\varrho}{\varrho+\delta}} = \frac{\varrho+\delta}{(\varrho+\delta)+\theta},$$

we have $E_i \sim \text{Exp}(\varrho+\delta)$ which can be sampled directly.

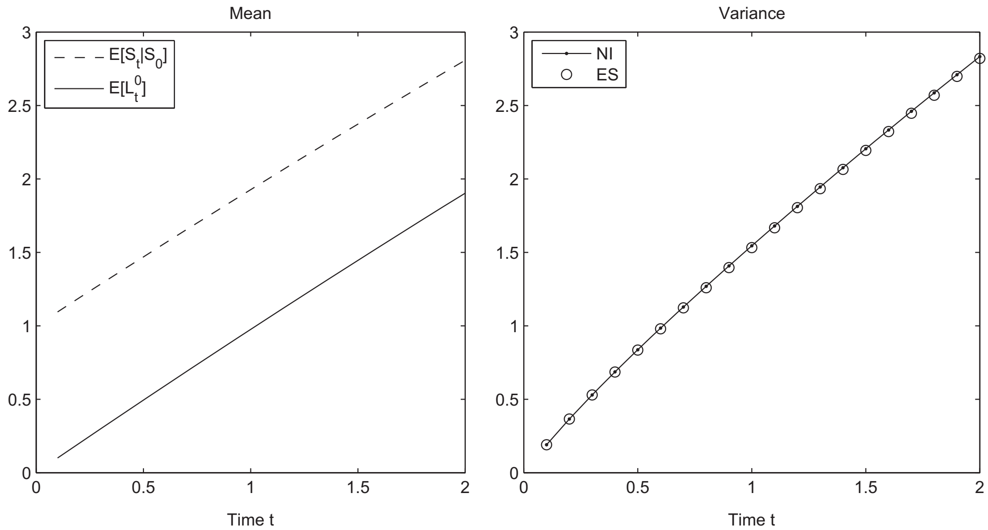

Figure 3. Means and variances of S_t and L_t^0 for the gamma case with $(\alpha, \beta) = (2, 2)$; the associated detailed numerical results are reported in Table 1.

5.2. Example: Gamma shot-noise process

For a Gamma distribution (including chi-squared distribution as a special case), i.e. $\tau_i \sim \Gamma(\alpha, \beta)$ with density function

$$p(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}, \qquad u > 0,$$

where $\alpha, \beta > 0$ are the shape and rate parameters, respectively, we have the mean is $\gamma_1 = \alpha/\beta$ and the Laplace transform

$$\hat{p}(\theta) = \left(\frac{\beta}{\beta + \theta}\right)^\alpha.$$

With the exact simulation via Algorithm 4.1 and using the parameter setting $(\alpha, \beta) = (2, 2)$, the associated conditional means and variances are plotted in Figure 3 and reported in Table 1. Each point in the variance plots is estimated by the exact simulation (ES) via Algorithm 4.1 with 10^7 replications. For implementing Algorithm 4.1, since a gamma distribution after the exponential tilting (4.21) is still a gamma distribution, i.e.

$$\hat{f}_{E_i}(\theta) = \frac{\hat{p}(\theta + \delta)}{\hat{p}(\delta)} = \frac{\left(\frac{\beta}{\beta+\theta+\delta}\right)^\alpha}{\left(\frac{\beta}{\beta+\delta}\right)^\alpha} = \left(\frac{\beta + \delta}{\beta + \delta + \theta}\right)^\alpha,$$

we have $E_i \sim \Gamma(\alpha, \beta + \delta)$ which can be sampled directly.

5.3. Example: inverse Gaussian shot-noise process

We can also make an immediate application to modelling credit defaults: N_t can be used for modelling the arrivals of credit defaults (of e.g. corporate bonds) in a large credit portfolio, X_i is the loss of the ith credit default, then, L_t^0 can be interpreted as the present value of the total loss of this credit portfolio within the time period of $[0, t]$. Our results for the moments of L_t^0 provided in Section 4.2. tell people the moments of the portfolio loss, which could be useful for credit risk management and

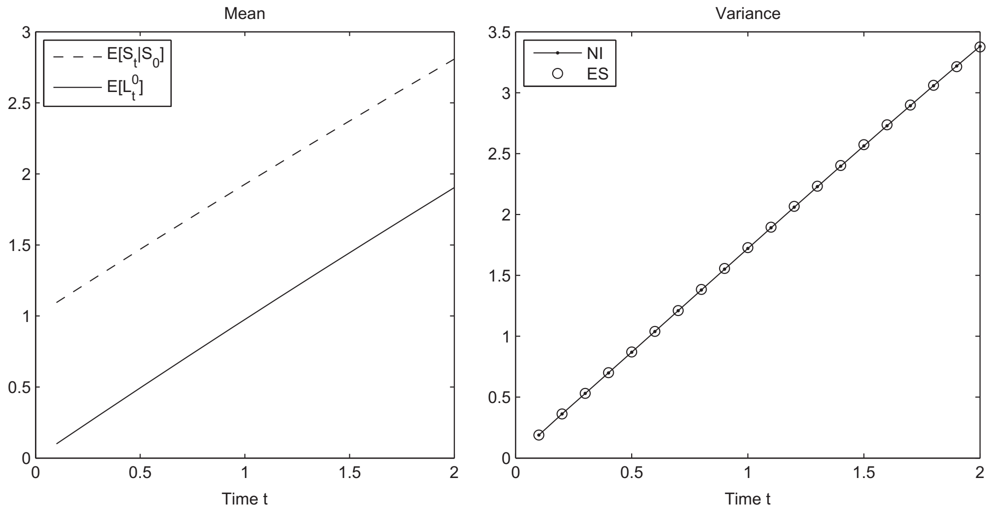

Figure 4. Means and variances of S_t and L_t^0 for the inverse Gaussian (IG) case with $(a, b) = (1, 1)$; the associated detailed numerical results are reported in Table 1.

measurement. U_t is the time elapsed since the last default occurred[2]. For numerical illustration, we assume the interarrival times between two successive credit defaults follow an inverse Gaussian, i.e. $\tau_i \sim \mathrm{IG}(a, b)$ with density function

$$p(u) = \frac{a}{\sqrt{2\pi u^3}} e^{-\frac{(a-bu)^2}{2u}}, \qquad u, a, b > 0,$$

where the mean is $\gamma_1 = a/b$ and the shape parameter is a^2, we have the Laplace transform

$$\hat{p}(\theta) = e^{-\left[\sqrt{2\theta+b^2}-b\right]a}.$$

With the exact simulation via Algorithm 4.1 and using the parameter setting $(a, b) = (1, 1)$, the conditional means and variances of S_t and the present value of credit portfolio losses L_t^0 are plotted in Figure 4 and reported in Table 1. Each point in the variance plots is estimated by the exact simulation (ES) via Algorithm 4.1 with 10^7 replications. For implementing Algorithm 4.1, since an IG distribution after the exponential tilting (4.21) is still an IG distribution, i.e.

$$\hat{f}_{E_i}(\theta) = \frac{\hat{p}(\theta+\delta)}{\hat{p}(\delta)} = \frac{e^{-\left[\sqrt{2(\theta+\delta)+b^2}-b\right]a}}{e^{-\left[\sqrt{2\delta+b^2}-b\right]a}} = e^{-\left[\sqrt{2\theta+(2\delta+b^2)}-\sqrt{2\delta+b^2}\right]a},$$

we have $E_i \sim \mathrm{IG}\left(a, \sqrt{2\delta+b^2}\right)$ which can be sampled directly.

[2]Since the 2007 financial crisis, default rates of corporate bonds have decreased, as the world economy has emerged from the global financial crisis (GFC) with improving market conditions. However, default rates going forward are dependent on the progress of world economic recovery and growth, as well as oil and commodity prices, fiscal and monetary policy and interest rate fluctuations. Hence, in specific situations like 2007-2008 GFC, the time elapsed since the last default occurred could be an important parameter in credit default modelling. The properties of renewal shot-noise processes and the results newly found in this paper could be also appropriate for modelling credit risk.

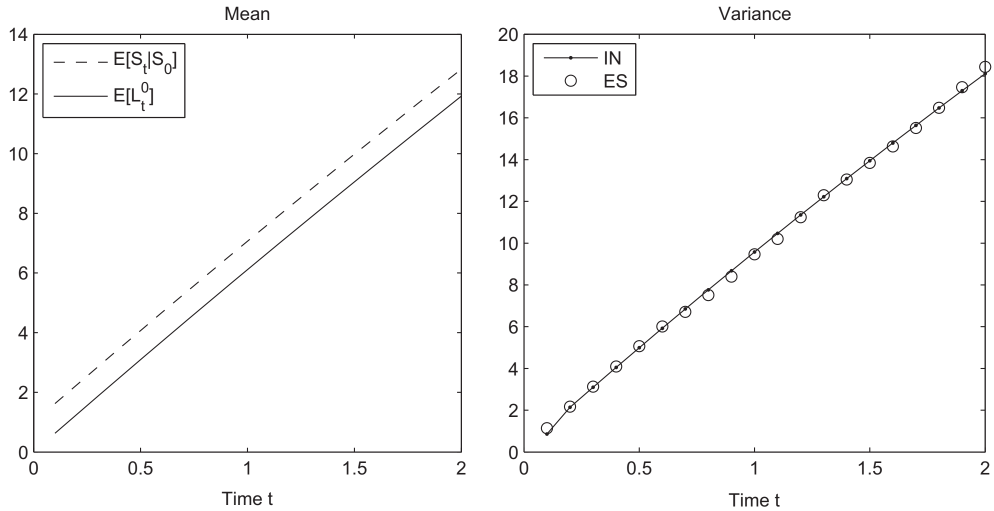

Figure 5. Means and variances of S_t and L_t^0 for the folded normal (FN) case with $(\mu, \sigma) = (0, 0.2)$; the associated detailed numerical results are reported in Table 1.

5.4. *Example: folded normal shot-noise process*

Queues and related models are important in solving many complex reliability problems. The renewal shot-noise process and its variations can be considered to deal with the expected busy periods in terms of queuing system and the virtual waiting times of customers, etc. Due to the similar nature of cashflow structure, our results could also be applied to reliability modelling. N_t accounts the total number of failures of machine components up to time t. X_i is the individual cost of the ith failure, then, L_t^0 can be interpreted as the present value of the total cost within the time period of $[0, t]$. For reliability modelling, we take the *folded normal* distribution (including the *half-normal* distribution as a special case) as an example for modelling the interarrival times of two successive failures, i.e. $\tau_i \sim \text{FN}(\mu, \sigma)$ with density function

$$p(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \left[e^{-\frac{(u-\mu)^2}{2\sigma^2}} + e^{-\frac{(u+\mu)^2}{2\sigma^2}} \right] \qquad u \geq 0,$$

where $\mu, \sigma > 0$, we have the mean

$$\gamma_1 = \sqrt{\frac{2}{\pi}} \sigma e^{-\frac{\mu^2}{2\sigma^2}} + \mu \left[1 - 2\Phi\left(-\frac{\mu}{\sigma} \right) \right],$$

and the Laplace transform

$$\hat{p}(\theta) = e^{\frac{\sigma^2}{2}\theta^2 - \mu\theta} \left[1 - \Phi\left(-\frac{\mu}{\sigma} + \sigma\theta \right) \right] + e^{\frac{\sigma^2}{2}\theta^2 + \mu\theta} \left[1 - \Phi\left(\frac{\mu}{\sigma} + \sigma\theta \right) \right]. \tag{5.1}$$

Each point in the variance plots is estimated by the exact simulation (ES) via Algorithm 4.1 with 10^7 replications. For implementing Algorithm 4.1, since an folded normal distribution after the exponential tilting (4.21) is unknown, we need the *acceptance/rejection (A/R) scheme* of Algorithm 2, where $\tau_i \sim \text{FN}(\mu, \sigma)$ in Step 1 can be simply sampled via

$$\tau_i \stackrel{\mathcal{D}}{=} |\mu + \sigma V|, \qquad V \sim \mathcal{N}(0, 1).$$

With the exact simulation via Algorithm 4.1 and using the parameter setting $(\mu, \sigma) = (0, 0.2)$, the conditional means and variances of S_t and the present value of total cost L_t^0 are plotted in Figure 5 and reported in Table 1.

6. Concluding remarks

We have mainly studied the Laplace transforms of the conditional and asymptotic moments for renewal shot-noise processes and discounted aggregate claims. A very efficient and general simulation algorithm has been developed for estimating the second conditional moments, and it has been compared with the alternative method of numerical inversion. For applications to the net premium calculation in insurance as well as credit risk and reliability modelling, the first conditional moments and variances for four different distributions of interarrival times have been computed, respectively. In fact, renewal shot-noise processes and the properties found in this paper could be also applicable to a wide range of other fields such as queueing, financial transaction data, computer networks, inventories and storage system, etc., and we leave them as further research.

Acknowledgements

The authors would like to thank all reviewers for very helpful and constructive comments and suggestions. They also thank Yan Qu for helping the implementation of numerical inversion of Laplace transforms. The corresponding author Hongbiao Zhao would like to thank Yao Xiao for a very careful read and acknowledges the financial support from the *National Natural Science Foundation of China* (#71401147,#91546202) and the research fund provided by Shanghai University of Finance and Economics.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the National Natural Science Foundation of China [grant number 71401147], [grant number 91546202]; the research fund provided by Shanghai University of Finance and Economics.

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Appendix 1. Numerical inversion of Laplace transform

There are numerous different schemes available in the literature for Laplace numerical inversions. The aim of using numerical inversion in this paper is for comparing and validating our newly developed exact simulation algorithms. As finding optimal schemes for numerical inversions is not our main focus here at the current stage, we thereby adopt conventional ones. For the first three cases in Section 5, i.e. exponential, gamma and inverse Gaussian cases, which are easier, we simply apply the classical *Talbot algorithm* (Abate & Whitt 2006, Section 6, p. 416). It works very well by using the existing package of MatLab codes euler_inversion_sym.m available at MathWorks. For the folded normal case, which is more complicated due to the special function $\Phi(\cdot)$ in the Laplace transform $\hat{p}(\theta)$ specified in (5.1), we develop our own codes based on the *Euler algorithm* (Abate & Whitt 2006, Section 5, p. 415–416) with the aid of existing C++ package RcppFaddeeva from CRAN that can deal with the function $\Phi(\cdot)$ for complex values. Both algorithms involve *tuning (or scaling) parameters*, which are simple deterministic functions of positive integer M, the number of terms for approximating the infinite summation, that controls the associated truncation errors. More precisely, as illustrated in Abate & Whitt (2006), for a given Laplace transform $\hat{f}$ of a function f, i.e.

$$\hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\mathrm{d}t,$$

the underlying function f can be approximated by

$$f(t) \approx f_n(t) \equiv \frac{1}{t}\sum_{k=0}^{n} \omega_k \hat{f}\left(\frac{\alpha_k}{t}\right), \qquad t > 0, \tag{A1}$$

where the *nodes* α_k and *weights* ω_k are the associated *tuning (or scaling) parameters*, which are complex numbers, and depend on n but not on the transform $\hat{f}$ or the time argument t. These tuning parameters are specified differently by Talbot algorithm and Euler algorithm as follows:

- For the *Talbot algorithm* (Abate & Whitt 2006, Section 6, p. 416), the parameters in the framework (A1) are $n = M$, $\alpha_k = \delta_k$ and $\omega_k = \frac{2}{5}\gamma_k$, where

$$\delta_0 = \frac{2}{5}M, \qquad \delta_k = \frac{2}{5}k\pi\left[\cos\left(\frac{k\pi}{M}\right) + i\right], \qquad 0 < k < M,$$
$$\gamma_0 = \frac{1}{2}e^{\delta_0}, \qquad \gamma_k = \left[1 + i\frac{k\pi}{M}\left(1 + \cot^2\left(\frac{k\pi}{M}\right)\right) - i\cot\left(\frac{k\pi}{M}\right)\right]e^{\delta_k}, \qquad 0 < k < M,$$

  with $i \equiv \sqrt{-1}$.

- For the *Euler algorithm* (Abate & Whitt 2006, Section 5, p. 415–416), the parameters in the framework (A1) are $n = 2M$, $\alpha_k = \beta_k$ and $\omega_k = 10^{\frac{M}{3}} \eta_k$, where

$$\beta_k = \frac{\ln 10}{3} M + \pi ik, \qquad \eta_k \equiv (-1)^k \xi_k,$$
$$\xi_0 = \frac{1}{2}, \qquad \xi_k = 1, \quad 1 \le k \le M, \qquad \xi_{2M} = \frac{1}{2^M},$$
$$\xi_{2M-k} = \xi_{2M-k+1} + 2^{-M} \binom{M}{k}, \qquad 0 < k < M.$$

In our paper, we adopt $M = 64$ for the first three cases and $M = 6$ for the folded normal case, which determine the associated tuning parameters, respectively. From extensive numerical experiments, we do find that the estimation results from numerical inversion algorithms could be unstable for some sets of parameters. Indeed, it is a very typical problem for the numerical inversion approach, and it is why we shall advocate using our newly developed exact simulation approach as an alternative.

Appendix 2. Random variate generator for E_i

Algorithm B.1 (A/R Scheme for E_i): *For exactly sampling one random variable E_i in general:*

(1) *Generate a random variable E_e with density $p(u)$;*
(2) *Generate a uniformly distributed random variable $V \sim \mathcal{U}[0, 1]$;*
(3) *If $V \le e^{-\delta E_e}$, then, accept and set $E_i = E_e$; otherwise, reject and go back to Step 1.*

***Proof*:** To exactly sample a random variable E_i with the density function (4.21), we adopt the acceptance/rejection (A/R) scheme with the envelop density function $f_{E_e}(t) = p(t)$. Then, we can easily find the smallest possible constant K such that

$$\frac{f_{E_i}(t)}{f_{E_e}(t)} = \frac{e^{-\delta t}}{\hat{p}(\delta)} \le K.$$

Hence, we have $K = 1/\hat{p}(\delta)$, and the acceptance level

$$\frac{f_{E_i}(t)}{Kp(t)} = e^{-\delta t}.$$

Note that, the probability of acceptance is $1/\hat{p}(\delta)$. □